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Abstract

This article studies the Nash equilibrium of a simultaneous game with two players: a dependent elderly person (the parent - he) and his caregiver (the child - she). Both are altruistic, can transfer money to each other, and can provide the parent with long-term care, the parent by purchasing formal care on the market and the child by providing unpaid informal care. The Nash equilibrium can take three different forms as regards money transfers: the parent (resp. the child) makes a money transfer to the child (resp. the parent) if he (resp. she) is sufficiently richer than his child (resp. than her parent), otherwise there is no money transfer. Money transfers are thus used by players to keep the distribution of the family wealth and long-term care efforts within a frame that is "acceptable" from the players' points of view, but which can be very unfair from a regulator's point of view. Analyzing how the Nash equilibrium is modified with marginal variations of the parameters yields surprising findings: the case can arise where a parent would rather a regulator taxed his income ex ante to enrich his child, or where a parent eligible for a lump sum public allowance would rather it was paid to his child instead of to himself. We show that the Nash equilibrium is generally not Pareto-efficient, except when the child makes a money transfer to her parent. Not all Pareto-efficient allocations can be decentralized through an ex ante public system of taxation/subsidies of long-term care efforts and ex ante lump sum transfers. To achieve Pareto-efficient allocations that can be decentralized, the regulator should subsidize informal care to reduce its opportunity cost.

Keywords: long-term care, family transfers, altruism, Nash equilibrium, Pareto-efficient allocations, second-best allocations, taxation/subsidies of long-term care.

J.E.L. Classification: D1, D6, H21, H31, I1.

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1 Introduction

With their ageing populations, industrialized countries are experiencing a growing need for long-term care (hereafter LTC). LTC is defined by the OECD as "*a range of services for people who are dependent on help with basic activities of daily living (ADL) over an extended period of time. Such activities include bathing, dressing, eating, getting in and out of bed or chair, moving around and using the bathroom. These LTC needs are due to long-standing chronic conditions causing physical or mental disability*" (Huber and Hennessy, 2005). Research in several countries shows that the costs of LTC can be high, even for people with lower-level needs (Muir, 2017). These observations raise questions of LTC provision and funding.

As public LTC transfers remain relatively small in most countries,[1] and as the private insurance sector provides only limited options for covering LTC costs,[2] the first line of support for dependent elderly people is often their family, who provide unpaid care, referred to as informal care. Informal caregiving represents an important component of care for the elderly. Informal caregivers are typically daughters or spouses who spend time looking after a parent/husband when they could be using that time for paid work. Although informal care comes free of charge for the care receiver, the potentially very high opportunity cost for caregivers is now well recognized and has been assessed by researchers (Rodrigues et al., 2013; Van Houtven et al., 2013). In particular, the need to give up work can result in marked loss of income.

The other principal source of help for people with LTC needs is formal care, provided by professional carers, either at home or in an institution, and often paid for from savings made by the elderly. For persons with high levels of dependency, the cost of formal care may represent a heavy burden, putting them at risk of poverty.

Members of families with a dependent parent thus face a twofold challenge: they must meet the needs of the dependent parent through the provision of formal and informal LTC, while managing the associated financial burden. This second challenge may involve private money transfers from one family member to another. The possibility that a child may provide informal care in exchange for *inter vivos* monetary transfers or a bequest has long been debated in the literature and often modeled (since Bernheim et al., 1985). However, in reality, it also happens that a child concerned about the well-being of her parent, but striving to meet her other family and occupational constraints, provides monetary assistance on top of informal care. It can also be the case that a parent who transfers money to his child does not especially expect informal care in return, but purchases formal care on the market to lighten the burden on his child.

[1] De Donder and Leroux (2017) give several theoretical reasons for why there is so little social LTC transfer in most countries.

[2] Cremer et al. (2012) review the different explanations for the LTC insurance puzzle found in the literature: excessive cost (Brown and Finkelstein, 2007), reinforced by the presence of asymmetric information (Finkelstein and McGarry, 2006), unattractive rules of reimbursement (Cutler, 1993), crowding out by the family (Pauly, 1990), crowding out by the State (Norton, 2000; Brown et al., 2007), state-dependent utility (Finkelstein et al., 2013), myopia or ignorance (Finkelstein and McGarry, 2006 and Boyer et al 2017).

Dealing with the dependency of a parent in a family thus involves the *simultaneous* use of both formal and informal care and potentially of money transfers in both directions, and the reasons for these decisions can vary widely depending on the situation (there is no *a priori* reason to favor any one hypothesis for the motivation of family members, e.g. altruism, exchange or reciprocity).

The present paper sought specifically to investigate what guides the choice of families in the use of formal and informal care and within-family monetary transfers. We wanted to know how family members combined formal and informal care and money transfers both to meet the needs of the dependent parent and to manage the financial burden that the provision of LTC represents. For that purpose, we studied the Nash equilibrium of a simultaneous game with two players: a dependent elderly person (the parent - he) and his caregiver (the child - she). Both are altruistic – i.e. concerned with the other's well-being –, can make money transfers to each other and can provide the parent with LTC: the parent by purchasing formal care on the market and the child by providing unpaid informal care.

In the first part of our analysis, we show that according to the values of the parameters (especially the initial distribution of the family's wealth among its members), the Nash equilibrium can be of three different kinds as regards money transfers. There is no money transfer at equilibrium if the family's wealth is distributed in an acceptable manner between the family members, and the parent (resp. the child) makes a positive money transfer to the child (resp. the parent) if he (resp. she) holds a sufficiently large share of the family's wealth. The two LTC efforts depend on the distribution of the family's wealth at equilibrium. It is thus possible that two different situations as regards the *initial* distribution of the family's wealth (but identical as regards altruism and the family's total wealth) lead to the same LTC efforts and the same *final* distribution of total wealth. In a sense, in this non-cooperative framework, money transfers are used by the family members to redress inequalities (in terms of wealth and LTC provision) that are "unacceptable" to them. However, this use of money transfers can still lead to extreme situations that may seem unfair to a regulator, if one of the two players is much more altruistic than the other. For example, a child who is very altruistic may devote almost all her wealth to providing care for her parent, and her final net wealth will tend towards zero at Nash equilibrium: she willingly sacrifices her well-being to that of her parent. At the other extreme, a parent who is very altruistic may give all his wealth to his child and receive hardly any LTC.

We go on to analyze how the values of the equilibrium variables (LTC efforts and transfers) are modified by a marginal variation in each parameter. Some of our comparative statics results are surprising. For example, an increase in a child's altruism may reduce her parent's purchase of formal LTC when there is no money transfer or when the parent makes a positive money transfer at equilibrium. However, it may increase the parent's purchase of formal LTC when the child makes a positive money transfer at equilibrium. A more altruistic parent may then increase the amount of his money transfer and may reduce his purchase of formal LTC (while his child increases her informal LTC effort) if he makes a positive money transfer at equilibrium, but his degree of altruism has no effect on the equilibrium otherwise. It follows from these results that the parent's well-being is not always an increasing function of the initial share of the family's wealth that he holds. It may therefore happen that he would rather the government took some of his wealth ex ante and gave it to his child. Similarly, if he is eligible for an exogenous public lump sum allowance, the parent might prefer that this aid be paid directly to his child rather than to himself.

Finally, we focus on efficiency. We determine the set of Pareto-efficient allocations and analyze whether the Nash equilibrium is Pareto-efficient or not. First, we find that the Nash equilibrium is Pareto-efficient only if the child makes a positive money transfer to her parent. We then show that it is not possible to decentralize as Nash equilibria all Pareto-efficient allocations through a system of taxes or subsidies on LTC efforts and ex ante exogenous lump sum transfers. But when it is possible, the regulator should subsidize informal care, so that the opportunity cost of informal care for the child is lowered (and the parent's formal care effort is not distorted). This result is consistent with the idea that the child does not provide enough informal LTC at (inefficient) Nash equilibrium (except of course when she is sufficiently rich to make a positive money transfer to her parent at equilibrium). The reason why some Pareto-efficient allocations cannot be decentralized is that they are associated with a distribution of the family's wealth that is not acceptable to its members (because the share of the family's wealth received by one of the members is perceived as too high). In this case, there is a conflict between efficiency and equity: the regulator must give up efficiency and either tax the LTC efforts (so that the child's net wealth will be relatively high) or subsidize them (so that the child's net wealth will be relatively low).

The paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model. Section 4 characterizes the Nash equilibrium. Section 5 is dedicated to comparative statics. In Section 6, we focus on efficiency and on the feasibility of decentralizing Pareto-optimal allocations. We conclude in Section 7. All omitted proofs are reported in the appendix.

2 Related literature

There is a theoretical literature on intrafamily decisions starting with Becker's model of the family (Becker, 1974). A central aspect of this literature consists in analyzing the motivations for income transfers and provision of service from one family member to another (generally an adult child and her elderly parent). Depending on the focus of the papers, models vary in several aspects: which family members are altruistic or participate in the decision-making process, what types of care are considered, whether cash transfers may be made between family members, and by what mechanisms parent and child interact. To the best of our knowledge, no work before ours has studied the Nash equilibrium of a simultaneous game with double-sided altruism, in which four endogenous variables are subject to a decision: informal care provided by the child and formal care purchased on the market by the parent, plus money transfers in both directions.

The first models (Barro, 1974; Becker, 1974, 1991) – referred to as "pure one-sided altruism" models (Laferrère and Wolff, 2006) – consider that the parent is the only family member who is altruistic. This means that the child's well-being is part of the parent's utility. The parent leads the game and the child passively accepts the parent's transfer (which cannot be negative). Both are endowed with an exogenous income. The main result of this standard type of model is known as the *neutrality property*: when the parent makes a positive transfer to the child, a small change in the income distribution among family members does not modify their consumption because any such change will be offset by a change in the parent's transfer. The occurrence of a positive transfer and its size increase with the parent's income and decrease with the child's income. This is why money transfers are said to reduce inequality between individuals linked by altruistic relations. This neutrality property is actually the foundation of Becker's "Rotten Kid" theorem whereby if a family has an altruistic head (typically a parent) who transfers family resources to another member (the child), no matter how selfish that other member is, he/she has an incentive to maximize total family income. This ensures an efficient outcome and explains why a public transfer is said to *crowd out* family transfer in that case.

One of the first extensions to the pure one-sided altruism model was to relax the assumption of "pure" altruism. A parent's altruism is said to be "impure" when the parent's utility is directly and positively influenced by a commodity (or a "merit good") provided by the child, typically informal care. Providing this commodity is costly to the child as it requires an effort. The parent can foster his child's effort by making her a money transfer (*inter vivos*, e.g., Cox (1987), or in the form of a bequest, e.g., Bernheim et al. (1985), which is the same in a model with one period). These types of models are often called "exchange" models as opposed to models of altruism.[3] Bergstrom (1989) has shown that the presence of this second commodity (in addition to money) invalidates the neutrality property (and the Rotten Kid theorem) if the parent cannot commit himself to a money transfer as a function of the child's informal care provision (or a "bequest rule") before the child chooses the level of informal care. Compared to models of perfect altruism like Becker's, exchange models thus introduce the possibility of an inefficient outcome and restore the usefulness of thinking about public policies to enhance efficiency in a normative perspective.

[3] Sloan et al. (1997) do not find empirical support for strategic bequests as a motive for caregiving.

In a paper taking a normative approach, Cremer and Roeder (2017) studied a model with purely selfish children and purely altruistic parents, in which neither side can make a credible commitment. They show that children do provide some informal care to their parents so long as they can expect a bequest, but that the level of care provided is too low to be efficient in the *laissez-faire* subgame perfect equilibrium. They study the design of the public policies that could improve the provision of informal LTC and thus prove that the crowding out of family exchanges by public LTC benefit is not a problem in this context.

Cremer et al. (2016) are also interested in public policies able to improve efficiency (which is not attained at equilibrium) in a similar setting in which both parents and children are altruistic (parents perfectly and children imperfectly). They first show that decentralizing the first-best solution would require implementing tax on bequests and lump-sum transfers conditionally on the parent's health status, which is hardly feasible in reality. They therefore focus on second-best instruments: linear (proportional) taxes on bequests and children's labor earnings to finance a uniform LTC benefit.

Pestieau and Sato (2008) study the optimal design of a LTC policy in a setting where the parent's altruism toward his child is assumed away and the altruistic child can help her parent either in time (by providing informal care) or in money. Before knowing whether he will be dependent or not, the parent can make a money transfer to his child or purchase a private LTC insurance. If the parent purchases a private insurance ex ante, he does not receive any help from his child once he becomes dependent. They consider a number of parent-child pairs. Children have different market productivity. They first show that in the absence of public policy, children with low market productivity help their dependent parent with time, children with middle incomes let their parent resort to private insurance (and receive no gift ex ante) and high-income children provide financial assistance. They then introduce the possibility for a utilitarian government to provide public nursing homes and show that the choice of private insurance and public nursing home is dichotomous for parents of middle-productivity children (this means that private insurance is a substitute for public nursing homes, but not for children's assistance). When heterogeneity in parents' wealth is taken into account, it might be socially desirable to choose such a policy to induce rich parents of middle-productivity children to purchase private insurance and poor parents to resort to public nursing.

Whereas in Pestieau and Sato (2008) private insurance and filial assistance are mutually exclusive, Courbage and Zweifel (2011) consider a model in which a parent decides to purchase an LTC insurance while his child provides informal care. The viewpoint of the paper is positive and the parent and the child interact under the guise of a non-cooperative simultaneous game. By providing informal care, the child can lower the probability of nursing home use. For the parent, purchasing a private insurance does not affect the probability of living in a nursing home, but allows him to maintain a better level of consumption if he has to pay for a nursing home (although it reduces his consumption if he does not live in a nursing home). After showing that the parent's purchase of LTC coverage and the child's provision of informal care are strategic substitutes, the authors focus on non-equilibrium outcomes, associated with zero effort on the part of the child combined with the maximum possible amount of LTC purchased by the parent. They show how changes in exogenous variables affect the probability of these unstable outcomes. In particular, they show that the lower the parent's initial wealth, the more a decrease in LTC subsidy and a decrease in his child's expected inheritance are likely to increase the occurrence of non-equilibrium outcomes, hence an increase in the probability of nursing home use.

Finally, Courbage and Eeckhoudt (2012) consider the case in which the child makes the decision regarding both the level of LTC insurance and the level of informal care. They show that these two variables can be either strategic complements or substitutes depending on the degree of altruism shown by the child and the parent. In particular, they are strategic complements when the child is altruistic.

3 The model

We consider a family consisting of two agents, a dependent parent and a child respectively indexed 1 and 2. Each one's wealth is exogeneous and denoted by w_1 and w_2. We denote by $\overline{w}$ the total wealth of the family, such that $\overline{w} = w_1 + w_2$. The parent needs help to carry out his daily living activities. This help can be either provided by the child as informal care e_2, or purchased by the parent as formal care, for a total expense e_1. It is assumed that $e_1, e_2 \in [0, \overline{e}]$ with $\overline{e} > \overline{w}$.

The parent's ability to live his daily life can be expressed as a production function $h = h(e_1, e_2)$ increasing and concave in the amounts of formal and informal care received.

Purchasing formal care decreases the parent's wealth w_1 by e_1. The parent may make a cash transfer t_1 to the child and he may receive one – t_2 – from the child, with $t_1, t_2 \in [0, \overline{w}]$. The child incurs an opportunity cost of providing informal care e_2. The parent's utility, which depends on his utility of net wealth and his ability to carrying out daily living activities, has the additively separable form

$$u(w_1 - e_1 - t_1 + t_2) + h(e_1, e_2). \tag{1}$$

The individual utility of the child is

$$v(w_2 - e_2 + t_1 - t_2). \tag{2}$$

Both agents are assumed to be altruistic, in the sense that they are both concerned with the other's utility. The parent's objective function, denoted by U, is thus

$$U(e_1, e_2, t_1, t_2) = u(w_1 - e_1 - t_1 + t_2) + h(e_1, e_2) + \alpha v(w_2 - e_2 + t_1 - t_2) \tag{3}$$

where $\alpha > 0$ is a coefficient measuring the parent's altruism. Symmetrically, the child's objective function is

$$V(e_1, e_2, t_1, t_2) = \beta(u(w_1 - e_1 - t_1 + t_2) + h(e_1, e_2)) + v(w_2 - e_2 + t_1 - t_2) \tag{4}$$

where $\beta > 0$ is a coefficient measuring the child's altruism. We assume that $\alpha\beta < 1$.

The game

We study the Nash equilibria of a game in which the parent chooses e_1 and t_1 while the child chooses e_2 and t_2. The choices are simultaneous, and must satisfy,

$$y_i \equiv w_i - e_i - t_i + t_j \geq 0 \text{ , for all } i \neq j \; \in \{1, 2\} \tag{5}$$

$$e_1, e_2 \in [0, \overline{e}] \text{ and } t_1, t_2 \in [0, \overline{w}] \, . \tag{6}$$

Assumptions and notations

We denote by λ_0 the initial parent's share of the family's total wealth ($\overline{w}$). We thus have $\lambda_0 \overline{w} \equiv w_1$. It follows that $(1 - \lambda_0)\, \overline{w} \equiv w_2$, where $(1 - \lambda_0)$ is the share of the total wealth held by the child at the beginning of the game.

The parent's and the child's utility of net wealth, respectively $u(.)$ and $v(.)$, are both increasing and concave and are two-times derivable. We also assume that $\lim_{y \to 0}(u'(y)) = \lim_{y \to 0}(v'(y)) = +\infty$.

As usual, h_i denotes the first-order derivative of h with respect to e_i and h_{ij} the second-order derivative with respect to e_i and e_j. To ensure the existence of an interior equilibrium, we assume that $\lim_{e_i = 0} h_i(e_1, e_2) = +\infty$ and $\lim_{e_1 = 0, e_2 = 0}(h_1(e_1, e_2) + h_2(e_1, e_2)) = +\infty$. Moreover, $\lim_{e_i = \overline{w}, e_j = 0} h_i(e_1, e_2)$ exists and is finite for $i \neq j = 1, 2$. The ratio $\dfrac{h_1}{h_2}$ is the marginal rate of technical substitution (hereafter MRTS) of e_1 for e_2. We assume that this ratio is decreasing in e_1 and increasing in e_2.[4]

With these assumptions, the players' strategy sets are compact and convex and the strategies have to belong to a set of constraints – defined by (5) and (6) above – which is convex and compact. The objective functions are continuous and concave, which ensures, according to Rosen (1965), that the Nash equilibrium resulting from this game exists and is unique.

[4] That is, $h_{12} \geq \max(\dfrac{h_2}{h_1} h_{11}, \dfrac{h_1}{h_2} h_{22})$.

4 Properties of equilibrium

The Nash equilibrium of the game played by the parent and the child is a strategy combination $S^* = (e_1^*, e_2^*, t_1^*, t_2^*)$ such that each family member's strategy is a best response to the other member's strategy. Applying the methodology developped in Rosen (1965) gives:

Proposition 1 *In equilibrium, one transfer at least equates zero ($t_1 t_2 = 0$). Also,*

(i) $h_1(e_1, e_2) = u'(w_1 - e_1 - t_1 + t_2)$ and $\beta h_2(e_1, e_2) = v'(w_2 - e_2 + t_1 - t_2)$;

(ii) according to the values of parameters,

- *either $t_1 = t_2 = 0$ and $\dfrac{h_1}{h_2} = \dfrac{\beta u'}{v'} \in [\alpha\beta, 1]$ (case 1),*
- *either $t_1 > 0, t_2 = 0$ and $\dfrac{h_1}{h_2} = \dfrac{\beta u'}{v'} = \alpha\beta$ (case 2),*
- *or $t_2 > 0, t_1 = 0$ and $\dfrac{h_1}{h_2} = \dfrac{\beta u'}{v'} = 1$ (case 3).*

Proposition 1 provides necessary and sufficient conditions for equilibrium. In equilibrium, according to the values of parameters, one of the three presented cases occurs (no transfer, positive transfer from parent to child, or positive transfer from child to parent).[5]

Conditions (i) and (ii) are the marginal conditions ensuring that the parent and the child have no incentive to change their strategy at equilibrium. Condition (i) corresponds to the first-order conditions with respect to formal and informal care efforts. More precisely, the first condition corresponds to the cancellation of the derivative of the parent's objective function U with respect to e_1 and the second one corresponds to the cancellation of the derivative of the child's objective function V with respect to e_2. Given the transfers, neither the parent nor the child wishes to provide another amount of care.

The derivative of the parent's objective function U with respect to the transfer t_1 $(-u' + \alpha v')$ depends on the parent's altruism and must equal zero for positive values of transfer t_1. The parent thus does not prefer a marginal increase or decrease in the transfer. If the parent pays no money transfer in equilibrium, the derivative must be non-positive. The same reasoning can be applied to the child. The derivative of her objective function V with respect to t_2 $(\beta u' - v')$ must be non-positive in the case of a nil transfer, and nil in the case of a positive transfer t_2. According to condition (ii), the case in which both transfers are positive is not possible because it is not possible to have at the same time $u' = \alpha v'$ and $\beta u' = v'$. Consequently, if the parent makes a positive transfer ($u' = \alpha v'$), then the child makes no transfer because $\beta u' = \alpha\beta v' < v'$ according to conditions (i) and (ii). The same reasoning applies to the child: if the child makes a positive transfer, then the parent makes no transfer because $u' = \frac{1}{\beta} v' > \alpha v'$ according to conditions (i) and (ii).

Combining these equalities and inequalities, we find that the equilibrium is such that the ratio of marginal utilities of net wealth ($\frac{u'}{v'}$) is proportional to the MRTS of e_1 for e_2 ($\frac{h_1}{h_2}$)

[5] LTC efforts and net wealths are obviously positive.

($\beta \frac{u'}{v'} = \frac{h_1}{h_2}$ at equilibrium). The MRTS of e_1 for e_2 ($\frac{h_1}{h_2}$) is then bounded by $\alpha\beta$ and 1. However, since the MRTS meets $\frac{h_1}{h_2} = \beta \frac{u'}{v'}$ according to condition (i), this implies that the ratio of marginal utilities $\frac{u'}{v'}$ also varies between two bounds that are independent of respective initial wealths w_1 and w_2. The two cases in which the ratio of marginal utilities equals one of these bounds correspond to the two types of equilibria with transfer.

We denote by λ the share of the family's total wealth owned by the parent after transfers, i.e.

$$\lambda\, \overline{w} \equiv w_1 - t_1 + t_2.$$

Let $(\widehat{e}_1(\lambda, \overline{w}, \beta), \widehat{e}_2(\lambda, \overline{w}, \beta))$ be the solutions to conditions (i) of Proposition 1 rewritten as follows:

$$u'(\lambda\, \overline{w} - e_1) = h_1(e_1, e_2) \tag{7}$$
$$v'((1-\lambda)\, \overline{w} - e_2) = \beta h_2(e_1, e_2). \tag{8}$$

We observe that $(\widehat{e}_1(.), \widehat{e}_2(.))$ could be defined for all values of $\lambda \in (0,1)$. Moreover, $(\widehat{e}_1(\lambda, .), \widehat{e}_2(\lambda, .))$ corresponds to the equilibrium efforts of a game in which no transfer is feasible and the initial distribution of wealth would be λ. We denote by λ^e the equilibrium distribution of wealth inside the family.

Of course, when no transfer is paid in equilibrium, $\lambda^e = \lambda_0$ and the levels of equilibrium LTC efforts are $(\widehat{e}_1(\lambda_0, .), \widehat{e}_2(\lambda_0, .))$. But more generally, we know from Proposition 1 that the equilibrium efforts will be $(\widehat{e}_1(\lambda^e, .), \widehat{e}_2(\lambda^e, .))$. They thus correspond to an equilibrium of a reference situation in which transfers are unfeasible. Hence, the equilibrium is fully described by the equilibrium distribution of wealth λ^e.

Proposition 2 *Given $\overline{w}$, α and β, we consider the set of equilibria obtained when $\lambda_0 \in (0,1)$. There are two bounds $\underline{\lambda}$ and $\overline{\lambda}$ with $0 < \underline{\lambda} < \overline{\lambda} < 1$ such that*

(i) $\lambda^e = \lambda_0$ if $\lambda_0 \in [\underline{\lambda}, \overline{\lambda}]$. The equilibrium corresponds to case (1) where there is no transfer.

(ii) $\lambda^e = \underline{\lambda}$ if $\lambda_0 \leq \underline{\lambda}$ and $\lambda^e = \overline{\lambda}$ if $\lambda_0 \geq \overline{\lambda}$. The equilibrium corresponds to case (3) (respectively case (2)) where the child (resp. the parent) makes a positive transfer if $\lambda_0 \leq \underline{\lambda}$ (resp. $\lambda_0 \geq \overline{\lambda}$).

(iii) $g(\lambda) \equiv \dfrac{h_1(\widehat{e}_1(.), \widehat{e}_2(.))}{h_2(\widehat{e}_1(.), \widehat{e}_2(.))}$ is decreasing and $\underline{\lambda} = g^{-1}(1)$ and $\overline{\lambda} = g^{-1}(\alpha\beta)$.

(iv) $\widehat{e}_1(.)$ is an increasing function of λ and $\widehat{e}_2(.)$ a decreasing one.

Proposition 2 shows that whether there is a transfer from parent to child, from child to parent or no transfer depends on the initial distribution of wealth between family members and their degrees of altruism. If the parent's share of total wealth is too large ($\lambda_0 > \overline{\lambda} \equiv g^{-1}(\alpha\beta)$), there is a transfer from parent to child. If the parent's share of total wealth is too small ($\lambda_0 < \underline{\lambda} \equiv g^{-1}(1)$), there is a transfer from child to parent. If the distribution of wealth between family members is balanced $\lambda_0 \in \left[\underline{\lambda}, \overline{\lambda}\right]$, there is no transfer at equilibrium. Table 1 below summarizes Proposition 2

No transfer if $\lambda_0 \in \left[\underline{\lambda}, \overline{\lambda}\right]$, i.e. $\lambda^e = \lambda_0$
Transfer from parent to child if $\lambda_0 > \overline{\lambda}$, i.e. $\lambda^e = \overline{\lambda}$
Transfer from child to parent if $\lambda_0 < \underline{\lambda}$, i.e. $\lambda^e = \underline{\lambda}$.

Table 1

In this game with altruistic but non-cooperative players, transfers are thus used by family members to narrow wealth disparities within the family. If the distribution of wealth is too unequal ($\lambda_0 > \overline{\lambda}$ or $\lambda_0 < \underline{\lambda}$), the wealthier family member makes a transfer to the less wealthy one to return to the first acceptable share of wealth ($\overline{\lambda}$ if $\lambda_0 > \overline{\lambda}$ or $\underline{\lambda}$ if $\lambda_0 < \underline{\lambda}$). In other words, disparities of wealth are bounded above and below at equilibrium: the parent always obtains a net wealth between $\underline{\lambda}\ \overline{w}$ and $\overline{\lambda}\overline{w}$ and the child always obtains a net wealth between $(1 - \overline{\lambda})\ \overline{w}$ and $(1 - \underline{\lambda})\ \overline{w}$.

The decisions on the levels of LTC efforts do not therefore depend on the original distribution of wealth but on the *new* distribution – i.e. the distribution after transfers – which depends on the family's total wealth. In other words, the levels of LTC efforts depend on the respective initial wealth of family members when there is no transfer at equilibrium, whereas they depend on the total wealth of the family in the two equilibria with a transfer.

We notice that in equilibria with transfer, the levels of efforts always correspond to those obtained in another equilibrium without transfer (that would occur with another initial distribution of wealth). Basically, transfers also help to keeping levels of effort inside bounds that depend on the degree of altruism of family members and of the family's total wealth. It follows that the levels of LTC efforts lie in a limited range that depends on the "acceptable" disparities of wealth within the family: $e_1 \in [\widehat{e}_1(\underline{\lambda}, .), \widehat{e}_1(\overline{\lambda}, .)]$ and $e_2 \in [\widehat{e}_2(\overline{\lambda}, .), \widehat{e}_2(\underline{\lambda}, .)]$.

Numerical example

To derive explicit solutions for e_1, e_2, t_1 and t_2, and take the analysis further, we now consider logarithmic functions for u, v and h and assume that $\forall y,\ u(y) = v(y) = \ln(y)$ and $h = \ln(e_1 e_2)$. The logarithmic functions used in this example fulfill the hypothesis of the general setting.

Applying Proposition 2, we obtain

$$\begin{aligned} e_1 &= \frac{\lambda^e\ \overline{w}}{2} \text{ and } e_2 = \frac{\beta\ (1 - \lambda^e)\overline{w}}{1 + \beta}; \\ \underline{\lambda} &= \frac{2\beta}{1 + 3\beta} \text{ and } \overline{\lambda} = \frac{2}{2 + \alpha + \alpha\beta}. \end{aligned}$$

The set of Nash equilibria when the initial share of family wealth held by the parent λ_0 changes can be represented graphically. It corresponds to segment $[AB]$ on Figure 1.

insert Fig.1 here

If λ_0 is in the range between $]\underline{\lambda}, \overline{\lambda}[$, then the couple of equilibrium LTC efforts corresponds to a point on segment $]AB[$. In this case there is no transfer at equilibrium (case 1). Each point of segment $]AB[$ corresponds to an equilibrium without transfer and can thus be associated with a unique value of λ_0. By contrast, several values of λ_0 lead to point A and point B. Specifically, for all $\lambda_0 \leq \underline{\lambda}$, the couple of equilibrium LTC efforts corresponds to point A. In this case either the child makes a positive money transfer to her parent if $\lambda_0 < \underline{\lambda}$ (case 3) or there is no transfer if $\lambda_0 = \underline{\lambda}$ (case 1). For all $\lambda_0 \geq \overline{\lambda}$, the couple of equilibrium LTC efforts corresponds to point B. In this case, either the parent makes a positive money transfer to his child if $\lambda_0 > \overline{\lambda}$ (case 2) or there is no transfer if $\lambda_0 = \overline{\lambda}$ (case 1).

Figure 1 shows that a larger money transfer from the child cannot be associated with a lower LTC effort on his part. The equilibria where he makes the most LTC effort are in A, and they are therefore also those where the child is richest and transfers the most. The same is true for the parent, since it is at point B that the parent makes the most effort and also transfers the most money.

It is noteworthy that while the upper bound of the acceptable range of wealth distribution $\overline{\lambda}$ depends (negatively) on α and β, the lower bound $\underline{\lambda}$ depends only (positively) on β (but does not depend on α). Therefore, the values of endogenous variables (levels of transfer and levels of LTC efforts) at equilibrium with a transfer from parent to child depend on both the parent's and the child's degree of altruism, whereas they depend only on the child's altruism at equilibrium with a transfer from child to parent. We also note that although α does not affect the value of efforts at equilibrium without transfer, it affects the likelihood of having such an equilibrium.

Do transfers between family members insure a "fair" distribution of wealth inside the family and "fair" LTC effort ?

In other words, do transfers protect against extreme situations in which one of the family members holds almost all the familily's wealth? The answer to this question is no, because according to the degree of the members' altruism, the set of Nash equilibria can be "trapped" inside bounds, enabling one of the family members to sacrifice themselves for the other.

For example, for given $\alpha\beta < 1$, if α tends to infinity and β tends to 0, it is easy to show with the logarithmic example that the bounds $\underline{\lambda}$ and $\overline{\lambda}$ tend to 0. At Nash equilibrium, the child then holds almost all the family's wealth, and LTC efforts are both very low. This corresponds to a situation in which the parent sacrifices himself for his child (in particular, he transfers almost all his wealth to his child however little he has).

If, for the same $\alpha\beta < 1$, β tends to infinity and α tends to 0, the bounds become $\underline{\lambda} = \frac{2}{3}$ and $\overline{\lambda} = \frac{2}{2+\alpha\beta}$. The parent then holds in equilibrium at least two thirds of the family's wealth and the child devotes what wealth she accrues, however little, to helping her parent. The child's consumption is almost nil since her net wealth $y_2 \equiv w_2 - e_2 - t_1 + t_2$ is between $\alpha\,\overline{w}$ (which tends to 0 as α tends to 0) and $\frac{\overline{w}}{1+3\beta}$ (which tends to 0 as β tends to infinity). This corresponds to a situation in which the child sacrifices herself for her parent.

5 How do altruism and wealth change the equilibrium allocation?

Until now we have discussed our results from the point of view of the distribution of wealth within the family and we have shown what happens when this distribution changes. Specifically, we have shown that different distributions of wealth could lead to Nash equilibria of different kinds (with or without money transfer). In this section, we will use comparative statics to show how a marginal change in individual wealth or family members' altruism may affect the value of endogenous variables *without changing the nature of the Nash equilibrium.*

5.1 Equilibrium with no transfer

In this subsection we analyze how e_1 and e_2 change in the equilibrium with no transfer when one parameter changes. With only two endogenous variables, this analysis corresponds to a standard comparative statics exercise, with two best-response functions given by the first-order conditions with respect to the parent's and the child's effort. We recall that we have in equilibrium without transfer

$$e_i = \widehat{e}_i(\frac{w_1}{w_1+w_2}, w_1+w_2, \beta) \text{ for } i = 1, 2.$$

where $\widehat{e}_1(.)$ and $\widehat{e}_2(.)$ are defined in (7) and (8). Based on this system of equations, the results of the comparative statics developed in the appendix are given in Table 2 below.

$t_1 = t_2 = 0$	de_1	de_2
dw_1	+	sign of h_{12}
dw_2	sign of h_{12}	+
$d\beta$	sign of h_{12}	+
$d\alpha$	0	0

Table 2

The main results of Table 2 lead to Proposition 3.

Proposition 3 *In the equilibrium with no transfer*

(i) the parent's (resp. the child's) effort increases with his (her) wealth, and increases with the child's (resp the parent's) wealth if and only if $h_{12} > 0$*;*

(ii) efforts do not vary when the altruism of the parent vary;

(iii) the child's effort increases when her degree of altruism increases;

(iv) the parent's effort increases when the child's degree of altruism increases if and only if $h_{12} > 0$*.*

In the equilibrium with no transfer, when w_1 (resp. w_2) increases, the representative curve of the parent's (resp. child's) best-response function shifts upward. This means that for any given values of e_2 (resp. e_1), the best response of the parent (resp. the child) in terms of e_1 (resp e_2) increases. As a consequence of strategic interactions between e_1 and e_2, the parent (resp. the child) will increase his (her) effort if e_1 and e_2 are strategic complements – that is, if $h_{12} > 0$ – and decrease his (her) effort if e_1 and e_2 are strategic substitutes – that is, if $h_{12} < 0$. This explains point (i) of Proposition 3.

Point (ii) in Proposition 3 is straightforward as no best-response function depends on the parent's degree of altruism.

Finally, an increase in the child's degree of altruism increases her effort for any given value of e_1 (point (ii) in Proposition 3) without changing the parent's best-response function. If e_1 and e_2 are strategic complements (resp. substitutes) the parent's effort finally increases (resp. decreases) in the equilibrium (point (iv) in Proposition 3).

Consequences for the parent's and the child's utilities

Proposition 3 produces interesting results regarding the variation of the parent's and the child's well-being at equilibrium as a consequence of variations in initial wealths and in the child's altruism. Recall that the parent's well-being at equilibrium is

$$(u + h)^e = u(w_1 - \widehat{e}_1) + h(\widehat{e}_1, \widehat{e}_2)$$

and the child's well-being at equilibrium is

$$(v)^e = v(w_2 - \widehat{e}_2).$$

Differentiating these equations and using Proposition 3, we see that an increase in the child's altruism β always benefits the parent but always burdens the child (because it increases the child's informal effort).

Moreover, an increase in the child's wealth w_2 always increases both the child's and the parent's well-being. On the one hand, the child prefers to be richer at equilibrium, even if she makes more informal efforts for her parent. On the other hand, the parent benefits from an increase in his child's wealth because he can receive more informal help from her. But we note that an increase in the parent's wealth *is not always beneficial to the child* because the child may make more effort when her parent is richer if the efforts of the parent and the child are strategic complements (i.e. if and only if $h_{12} > 0$).

Public policy implications

The fact that a marginal increase in the child's wealth could benefit both her parent and herself suggests that such a change could constitute a Paretian improvement of the equilibrium without transfer (all other things being equal). Accordingly, we have considered two types of public policies that move in this direction, namely: redistributing some income from parent to child (at the margin) and paying the child a (lump sum) care allowance.

Redistributing at the margin income from parent to child

Such a policy would increase the child's income and decrease the parent's income by the same amount ($dw_2 = -dw_1$). This is thus equivalent to (marginally) modifying the initial distribution of the total wealth of the family λ_0 (without changing the nature of the equilibrium which remains an equilibrium without transfer). We can easily show that the child always benefits from such a change as her utility function at equilibrium v^e decreases with λ_0. But the well-being of the parent at equilibrium, measured by $(u+h)^e$ is not a monotonic function with respect to λ_0. This means that the parent could prefer to be poorer relative to his child and benefit from more informal help from her. This can be easily demonstrated with our logarithmic example. In case 1, i.e. when $\lambda_0 \in \left[\underline{\lambda}, \overline{\lambda}\right] = \left[\frac{2\beta}{1+3\beta}, \frac{2}{2+\alpha+\alpha\beta}\right]$, the well-being of the parent at equilibrium is given by the function

$$(u+h)^e = \ln \beta \frac{(\lambda_0)^2(1-\lambda_0)\ (\overline{w})^3}{4+4\beta}.$$

This function increases with λ_0 if $\lambda_0 \leq 2/3$ and otherwise decreases. We also show that $\underline{\lambda} < 2/3$ and $2/3 \leq \overline{\lambda}$ if and only if $\alpha+\alpha\beta \leq 1$. This means that if the family members are not sufficiently altruistic, then the parent maximizes his well-being at equilibrium with $\lambda_0 = 2/3$. Hence if the share of the family's wealth owned by the parent is higher than 2/3, the parent would benefit from an exogenous transfer of his income to his child that would lead to $\lambda_0' = 2/3$. Since the child would also benefit from such a policy, this would constitute a Pareto improvement of the equilibrium.

Paying a (lump sum) care allowance to the child

Many countries provide financial assistance to families with a dependent elderly parent. Let us suppose that the government decides to pay a family a lump sum allowance δ. This allowance could be paid to the parent to help him purchase formal care services or to the carer, that is the child. But which is the better option ? In both cases, the allowance would increase the family's total wealth from $\overline{w}$ to $\overline{w}+\delta$. However, according to who receives it, the allowance would also change the distribution of the wealth measured by λ_0 differently. In particular, λ_0 increases if the parent receives δ and λ_0 decreases if the child receives δ. The above demonstration means that both the child and the parent would benefit from a lump sum allowance allocated to the child if $\lambda_0 > 2/3$ and if family members are not too altruistic $(\alpha(1+\beta) \leq 1)$. This result is very important for public policy because it means that a dependent elderly person eligible for an LTC allowance could prefer that this allowance be paid to his child instead of himself. This means that countries that have adapted their system of allocation of LTC allowances to allow the payment of an informal carer are exercising sound policy by making it possible to achieve Pareto improvements.[6]

5.2 Equilibria with transfer

In equilibrium with transfer from parent to child (case 2), the LTC efforts are

$$e_i = \widehat{e}_i(\overline{\lambda}(w_1 + w_2, \beta, \alpha), w_1 + w_2, \beta) \qquad \text{for all } i = 1, 2.$$

In equilibrium with transfer from child to parent (case 3), the LTC efforts are

$$e_i = \widehat{e}_i(\underline{\lambda}(w_1 + w_2, \beta), w_1 + w_2, \beta) \qquad \text{for all } i = 1, 2.$$

If we compare with the case with no transfer, we see that LTC efforts are now impacted by a change in one of the parameters w_1, w_2 and β both directly and through the change in the thresholds $\overline{\lambda}$ and $\underline{\lambda}$. We also note that the efforts will now be impacted (indirectly) by a change in the parameter α in the equilibrium with transfer from parent to child.

It is important to note that in an equilibrium with a transfer, the effect of a change in one parameter on the equilibrium value of efforts and transfer is more complicated to understand because the best-response strategy of the agent making the transfer cannot be reduced to one best-response function ; it is instead the combination of two best-response functions. The proof of Proposition 4 given in the appendix shows that we obtain the results presented in Table 3 below, where $^{(*)}$ means "if h_{12} is negative or nil or low".

[6] This is the case in France, for example, where legislation evolved a few years ago to allow elderly people eligible for personal allowance for autonomy (APA) to use it to pay for a relative to provide care.

$t_1 - t_2 \neq 0$	de_1	de_2	$dt_1 - dt_2$
dw_1	$+$	$+$	$+$
dw_2	$+$	$+$	$-$
$d\beta$ if $t_1^* > 0$	$-^{(*)}$	$+$	$+^{(*)}$
$d\alpha$ if $t_1^* > 0$	$-^{(*)}$	$+^{(*)}$	$+$
$d\beta$ if $t_2^* > 0$	$+$	$+$	$-$
$d\alpha$ if $t_2^* > 0$	0	0	0

Table 3

Proposition 4 summarizes what can be learned from Table 3 above.

Proposition 4 *If a transfer is made in equilibrium,*

(i) efforts increase with the family's total wealth;

(ii) when there is a transfer from parent to child (resp. from child to parent), the transfer increases when the parent's (resp. child's) wealth increases and decreases when the child's (resp. parent's) wealth increases;

(iii) when there is a transfer from child to parent, efforts and transfer increase with the child's altruism (but do not change with the parent's altruism);

(iv) when there is a transfer from parent to child, if h_{12} is negative or nil or low, more altruism leads to more effort from the child and more transfer from the parent but less effort from the parent.

Several comments are needed here.

First, as both efforts depend on the family's total wealth in equilibrium with transfer, the fact that an increase in either the parent's or the child's wealth leads to an increase in both efforts is not surprising (i). However, it is interesting to note that the consequence is that in an equilibrium with transfer, how the child's (resp. the parent's) effort changes when the parent's (resp. the child's) wealth increases no longer depends on the sign of h_{12}. Both efforts can then be considered as normal goods with respect to the family's total wealth in equilibria with transfer.

Second, (ii) means that in an equilibrium with transfer, if there is an increase in the wealth of the family member who is paying, the transfer increases, whereas it decreases if there is an increase in the wealth of the family member who receives the transfer. This result confirms that transfers are used by family members to redistribute wealth within the family: if one family member becomes richer, then that member gives more or receives less. It follows from the two preceding statements that an increase in the child's wealth is always beneficial to the well-being of the parent who gives less or receives more cash transfer (according to the nature of the equilibrium) and receives more informal help from his child.

Third, the effect of the parent's altruism on efforts and transfers is different according to the nature of the equilibrium. In particular, an increase in the parent's altruism has no impact on equilibrium when there is a transfer from child to parent (the same when there is no transfer). However, in the equilibrium with transfer from parent to child, it leads to an increase in the parent's transfer and a decrease in the parent's effort. In return, the child increases her effort. It is interesting to observe that when the parent is more altruistic, there is kind of "exchange" process in which the parent makes a bigger transfer to the child, who then makes more effort, while the parent makes less effort. We can show with our logarithmic example that an increase in the parent's altruism could be beneficial to the child's well-being but detrimental to the parent's well-being, since v^e increases with α and $(u+h)^e$ decreases with α when α is sufficiently high.[7]

Fourth, the effect of the child's altruism on efforts and transfers is different according to the nature of the equilibrium. In particular, the effect of the child's altruism on the parent's efforts depends on the sign of h_{12} when there is no transfer, whereas this is no longer the case in equilibria with transfer. An increase in the child's altruism increases the parent's effort in the equilibrium with transfer from child to parent (whatever the value of h_{12}) but decreases the parent's effort in the equilibrium with transfer from parent to child (when h_{12} is negative or null or sufficiently low). It is of note that in the case with transfer of the child, an increase in the child's altruism always increases the parent's well-being and decreases the child's well-being.

Effect of a redistribution of some income from parent to child and of paying a lump sum allowance to one of the family members

We conclude this discussion by stating that when there is a transfer at equilibrium, *redistributing at the margin income from parent to child* (as we envisaged in the equilibrium without transfer) will have no effect on equilibrium efforts or well-being of the family members since the family's total wealth remains unchanged (so long as this policy does not change the nature of the equilibrium). In the same way, if we consider an *exogenous lump sum allowance granted to the family*, the identity of the family member who receives it does not matter for the members' well-being or the level of LTC efforts since these depend only on the family's total wealth and altruism. Whoever the member is who receives this lump sum allowance, both the child and the parent will increase their effort and their well-being.

[7] In the logarithmic example, we have for the equilibrium with transfer from child to parent: $(u+h)^e = \ln \dfrac{\alpha\beta\ (\overline{w})^3}{(2+\alpha+\alpha\beta)^3}$ and $v^e = \ln \dfrac{\alpha\ \overline{w}}{(2+\alpha+\alpha\beta)}$. We show that $\dfrac{\partial(u+h)^e}{\partial\alpha} \geq 0$ if and only if $\alpha \leq \dfrac{1}{1+\beta}$.

6 Efficiency

In this section, we focus on efficiency. We first determine the set of Pareto-optimal allocations (6.1) and then analyze whether the Nash equilibrium belongs to it or not (6.2). We finally study how to achieve a chosen first-best or second-best situation through a public policy (6.3).

6.1 First best allocations

A feasible allocation is Pareto optimal if there is no other feasible allocation such that the utility of both family members ($u(.) + h$ and $v(.)$) increases, with the increase in the utility of one member being strict. A Pareto-optimal allocation (e_1, e_2, y_1, y_2) is then given by values of formal and informal efforts e_1 and e_2 and net wealth for the parent $y_1 \equiv w_1 - e_1 - t_1 + t_2$ and the child $y_2 \equiv w_2 - e_2 - t_2 + t_1$ solutions of

$$\max_{e_1,\, e_2 \in [0,\overline{e}],\, y_1 \geq 0, y_2 \geq 0} \quad u(y_1) + h(e_1, e_2)$$

$$v(y_2) \geq v(y^*)$$

$$y_1 + y_2 = \overline{w} - e_1 - e_2 \ ;$$

where y^* varies between 0 and $\overline{w}$ (the case $y^* = \overline{w}$ refers to a limit situation in which the parent gives all his wealth to the child and nobody makes any LTC effort). This program can be solved simply by observing that the child's wealth always saturates the constraint, so the set of Pareto-optimal allocations when y^* varies is given by:

$$u'(y_1) = h_1(e_1, e_2) = h_2(e_1, e_2) \tag{9}$$

$$y_1 = \overline{w} - e_1 - e_2 - y^* \text{ and } y_2 = y^* \in (0, \overline{w}) \,.$$

We will now use this result to evaluate the equilibrium in terms of Pareto efficiency.

6.2 Efficiency of the equilibrium

As a solution of a non-cooperative game, a Nash equilibrium is not generally expected to be Pareto-efficient. The presence of altruism in our model could influence this result. However, we can show that this is not totally true: when values of parameters lead to an equilibrium in which the child makes a positive transfer to her parent, the equilibrium corresponds to a Pareto-optimal allocation,[8] but other values of parameters lead to inefficient equilibria. In

[8] Note that if $t_2 > 0$ at equilibrium, e_1, e_2 and t_2 maximize $\beta u + v$ (there is no conflict of interest between the parent's choice and the child's choice of efforts at equilibrium).

other words, the equilibrium is efficient if and only if the child is richer enough than her parent and sufficiently altruistic.

Proposition 5 *If $\lambda_0 \leq \underline{\lambda}$, (the child makes a positive transfer to her parent), the equilibrium is Pareto-optimal. If $\lambda_0 > \underline{\lambda}$, it is possible to increase the child's and the parent's utility by decreasing the parent's LTC effort and by increasing the child's LTC effort.*

What follows illustrates the efficiency/inefficiency of the equilibrium when $u(.) = v(.) = \ln(.)$. We recall that with our logarithmic example we have in equilibrium

$$e_1 = \frac{\lambda^e\, \overline{w}}{2} \geq e_2 = \frac{\beta\,(1-\lambda^e)\,\overline{w}}{1+\beta} \text{ and } \lambda^e \in \left[\frac{2\beta}{1+3\beta}, \frac{2}{2+\alpha+\alpha\beta}\right],$$

and in this example, the efficiency condition (9) becomes

$$e_1 = e_2 = y_1 = \frac{\overline{w} - y^*}{3} \text{ and } y_2 = y^*,$$

so the set of Pareto efficient allocations is defined by: :

$$e_1 = e_2 = \frac{\lambda^* \overline{w}}{2} \text{ and } \lambda^* \in \left]0, \frac{2}{3}\right[,$$

with λ^* denoting the share of total wealth obtained by the parent in an optimum, that is $\lambda^* \overline{w} \equiv w_1 - t_1 + t_2$.

The example shows that Pareto-efficient allocations can be associated with a much poorer parent (gross and net wealth) than at equilibrium when $\lambda^* \in \left]0, \underline{\lambda}\right[$ but also with a richer parent than at equilibrium when $\lambda^* \in \left]\overline{\lambda}, \frac{2}{3}\right[$ (if $\overline{\lambda} < \frac{2}{3}$, which is the case when $\alpha + \alpha\beta > 1$). Figures 2a and 2b compare the optimal efforts to the equilibrium efforts. On these figures, the segment $[OP]$ defined by $e_2 = e_1 \in \left]0, \frac{\overline{w}}{3}\right[$ corresponds to the set of Pareto-efficient combinations of efforts when y^* varies between 0 and $\overline{w}$. Figures 2a and 2b also show that only one equilibrium is efficient. It corresponds to point A, as $[OP] \cap [AB] = \{A\}$.

insert Fig.2 here

Fig.2a ($\alpha + \alpha\beta > 1$)

Fig.2b ($1 \geq \alpha + \alpha\beta$)

On Figure 2a, we can see that there are Pareto-efficient situations in which the regulator requires less effort on the part of the two agents (the points close to point O). But conversely, there are also Pareto-efficient situations in which the regulator would require more effort on the part of both the parent and the child (the points close to point P).

Figure 2b shows that the parent can purchase too much formal LTC at Nash equilibrium compared to what is efficient (the set of equilibrium efforts of the parent is no longer included in the set of optimal efforts).

6.3 Second best allocations

We suppose here that the State levies a tax on LTC efforts while at the same time redistributing wealth within the family through exogenous transfers. This policy will change the equilibrium. We want to know whether a chosen (first-best) optimal situation can be decentralized as an equilibrium through this policy. If the decentralization of a chosen first-best optimum is not possible, what policy could achieve a second-best optimal equilibrium?

Let us consider a tax policy $(\delta_1, \delta_2, \tau_1, \tau_2)$, where δ_i $(i = 1, 2)$ denotes an exogenous cash transfer to agent i and τ_i denotes a tax on LTC effort e_1. This policy leads to an equilibrium allocation (e_1, e_2, y_1, y_2) where net wealths y_1 and y_2 are such that :

$$y_1 = w_1 - t_1 + t_2 + \delta_1 - (1 + \tau_1)e_1 \quad (10)$$

$$y_2 = w_2 - t_2 + t_1 + \delta_2 - (1 + \tau_2)e_2. \quad (11)$$

The State budget balance supposes that :

$$\delta_1 + \delta_2 = \tau_2 e_2 + \tau_1 e_1. \quad (12)$$

The set of feasible allocations at equilibrium induced by a policy (τ_1, τ_2) is defined below.

Definition 1 *One allocation (e_1, e_2, y_1, y_2) is feasible given the policy (τ_1, τ_2) if and only if it meets equations (13)(14)(15) and (16) below.*

$$y_1 + y_2 = \overline{w} - e_1 - e_2 \quad (13)$$

$$-(1 + \tau_1)\ u'(y_1) + h_1(e_1, e_2) = 0 \quad (14)$$

$$-(1 + \tau_2)\ v'(y_2) + \beta\ h_2(e_1, e_2) = 0 \quad (15)$$

$$\alpha \leq \frac{u'(y_1)}{v'(y_2)} \leq \frac{1}{\beta}. \quad (16)$$

The sum of (10) (11) and (12) gives (13). (14) ensures that the parent chooses the effort e_1 and (15) ensures that the child chooses the effort e_2 when they are faced respectively to taxes τ_1 and τ_2. Finally (16) ensures that neither the parent nor the child wishes to modify the net wealth received by the members of the family, i.e. the parent does not wish to increase his child's net wealth and the child does not wish to increase her parent's net wealth through private money transfers. In other words, any equilibrium allocations must satisfy these equations.[9]

We can easily note that taxes or subsidies that lead to a given feasible allocation are given by equations (14) and (15). Endogenous variables τ_1 and τ_2 and equations (14) and (15) can then be removed from the program above. A first-best Pareto-efficient allocation can thus be decentralized if it meets conditions (13) and (16).

It follows from this definition that the State has to subsidize the child's effort to obtain a given first-best Pareto-efficient allocations.[10] We recall that any efficient allocation $(e_1^*, e_2^*, \overline{w} - e_1^* - e_2^* - y^*, y^*)$ is defined by (9) so from (14) τ_1 must equal zero and from (15) and (16), we obtain that (9) implies that τ_2 is negative:

$$1 + \tau_2 \ = \frac{\beta \ u'(.)}{v'(.)} \leq 1.$$

Some first-best allocations cannot be decentralized. Applying (9) to the numerical example gives the set of efficient allocations

$$e_1^* = e_2^* = y_1^* = \frac{\overline{w} - y^*}{3} \text{ and } y_2^* = y^* \tag{17}$$

and (16) can be rewritten when the allocation is first-best efficient

$$\alpha \leq \frac{3y^*}{\overline{w} - y^*} \leq \frac{1}{\beta} \Leftrightarrow \frac{y^*}{\overline{w}} \in \left[\frac{\alpha}{3 + \alpha}; \frac{1}{1 + 3\beta} \right].$$

Proposition 6 below completes these results.

Proposition 6 *(i) The tax system needed to decentralize first-best Pareto efficient allocations meets $\tau_1 = 0$ and $\tau_2 < 0$.*

(ii) There are two thresholds $\underline{y}^*$ and $\overline{y}^*$ with $0 < \underline{y}^* < \overline{y}^* < \overline{w}$, such that the first-best Pareto-efficient allocation $(e_1^*, e_2^*, \overline{w} - e_1^* - e_2^* - y^*, y^*)$ can be decentralized if and only if $y^* \in \left[\underline{y}^*, \overline{y}^*\right]$.

(iii) For $u(.) = v(.) = \ln(.)$ we have $\underline{y}^* = \dfrac{\alpha \ \overline{w}}{3 + \alpha}$ and $\overline{y}^* = \dfrac{\overline{w}}{1 + 3\beta}$.

[9] All the allocations that are solutions of (13)(14)(15) and (16) can be an equilibrium in which the State proposes a tax system (τ_1, τ_2). To show that point, consider a given allocation $(\overset{\circ}{e}_1, \overset{\circ}{e}_2, \overset{\circ}{y}_1, \overset{\circ}{y}_2)$ that satisfies the above conditions and suppose that the State provides the transfer (δ_1', δ_2') such that $\delta_1' = -\overset{\circ}{t}_1 + \overset{\circ}{t}_2 + \delta_1$ and $\delta_2' = -\overset{\circ}{t}_2 + \overset{\circ}{t}_1 + \delta_2$. That is, the State gives to each family member the net transfer that member would receive in equilibrium. As the equilibrium allocation meets $\overset{\circ}{y}_1 = w_1 + \delta_1' - \overset{\circ}{e}_1$ and $\overset{\circ}{y}_2 = w_2 + \delta_2' - \overset{\circ}{e}_2$, the fiscal policy $(\delta_1', \delta_2', \tau_1, \tau_2)$ leads to an equilibrium where no transfer is paid in equilibrium. The new equilibrium leads to the same allocation $(\overset{\circ}{e}_1, \overset{\circ}{e}_2, \overset{\circ}{y}_1, \overset{\circ}{y}_2)$.

[10] This is consistent with the statement made above that the child does not exert enough long-term care effort in inefficient equilibria.

Point (ii) is shown in the appendix.

When first-best allocations cannot be decentralized, the second-best allocations are the solution to the following program (which gives the set of second-best allocations when $y^* \in [0, \overline{w}]$ varies):

$$\max_{e_i, \tau_i, y_i, i=1,2} \quad u(y_1) + h(e_1, e_2)$$
$$y_2 \geq y^*, \;\; y^* \in [0, \overline{w}]$$
$$st \; (e_1, e_2, y_1, y_2) \text{ satisfies (13)(14)(15) and (16).}$$

To further complete Proposition 6 we solve in the appendix the second-best program in the case of the logarithmic example. The allocation (e_1, e_2, y_1, y_2) solution of the program for the logarithmic example is

$$\left(\frac{\overline{w} - y^*}{3}, \frac{\overline{w} - y^*}{3}, \frac{\overline{w} - y^*}{3}, y^*\right) \text{ if } y^* \in \left[\frac{\alpha\, \overline{w}}{3+\alpha}, \frac{\overline{w}}{1+3\beta}\right] \tag{18}$$
$$\left(\frac{\overline{w} - (1+\beta)y^*}{2}, \frac{\overline{w} - (1+\beta)y^*}{2}, \beta\, y^*, y^*\right) \text{ if } y^* \in \left]\frac{\overline{w}}{1+3\beta}, \frac{\overline{w}}{1+\beta}\right[\tag{19}$$
$$\left(\frac{\alpha\overline{w} - (1+\alpha)y^*}{2\alpha}, \frac{\alpha\overline{w} - (1+\alpha)y^*}{2\alpha}, \frac{y^*}{\alpha}, y^*\right) \text{ if } y^* \in \left[\frac{\alpha\, \overline{w}}{3(1+\alpha)}, \frac{\alpha\, \overline{w}}{3+\alpha}\right] \tag{20}$$
$$\left(\frac{\overline{w}}{3}, \frac{\overline{w}}{3}, \frac{\overline{w}}{3(1+\alpha)}, \frac{\alpha\, \overline{w}}{3(1+\alpha)}\right) \text{ if } y^* \in \left]0, \frac{\alpha\, \overline{w}}{3(1+\alpha)}\right[. \tag{21}$$

Equation (18) corresponds to the set of first-best allocations described in equation (17), which can be decentralized with $\tau_1 = 0$ and $\tau_2 < 0$ as stated in Proposition 6. Equations (19), (20) and (21) refer to second-best allocations (allocations that cannot be decentralized as first-best optima). The tax systems to decentralize the allocations of (19), (20) and (21) are the following:

$$(19)\; \tau_1 = \tau_2 = \frac{(3\beta+1)\; y^* - \overline{w}}{\overline{w} - (1+\beta)\; y^*} > 0;$$

$$(20)\; \tau_1 = \frac{(3+\alpha)\; y^* - \alpha\, \overline{w}}{\alpha\, \overline{w} - (1+\alpha)\; y^*} < 0 \text{ and } \tau_2 = \frac{(1+\alpha+2\alpha\beta)\; y^* - \alpha\, \overline{w}}{\alpha\, \overline{w} - (1+\alpha)\; y^*} < 0;$$

$$(21)\; \tau_1 = \frac{-\alpha}{1+\alpha} < 0 \text{ and } \tau_2 = \frac{-1-\alpha+\alpha\beta}{1+\alpha} < 0.$$

The first best Pareto-optimal allocations that give too much net wealth to one of the agents cannot be decentralized. First-best Pareto-efficient allocations have been defined for $y^* \in [0, \overline{w}]$, but the first-best Pareto-efficient allocations that can be decentralized with a tax system are such that $y^* \in \left[\frac{\alpha\, \overline{w}}{3+\alpha}, \frac{\overline{w}}{1+3\beta}\right] \subset [0, \overline{w}]$. This is because family members define a set of "acceptable" distributions of wealth according to their degree of altruism and redistribute wealth between them if they find the initial distribution not "acceptable" to them.

Allocations such that $y^* \in \left[\frac{\overline{w}}{1+\beta}, \overline{w}\right]$ are not feasible with a tax system, either in a first-best or in a second-best; all the feasible allocations are such that $y^* \in \left[0, \frac{\overline{w}}{1+\beta}\right]$. This limits the (first-best and second-best) efficiency of a public policy if the child is very altruistic, since if β is very high (tends to infinity), $\frac{\overline{w}}{1+\beta}$ tends to 0, and very few allocations can ultimately be decentralized in a first or second-best.

However, the set of allocations that can be decentralized with a tax system allows more redistribution between the agents than equilibria without a tax system. We found that at Nash equilibrium, the net wealth of the child was $y_2 \in \left[\frac{\alpha\,\overline{w}}{2+\alpha+\alpha\beta}, \frac{\overline{w}}{1+3\beta}\right] \subset \left[0, \frac{\overline{w}}{1+\beta}\right]$.

Equation (19) refers to a situation in which the State wants the child to have a net wealth $\frac{\overline{w}}{1+3\beta} < y^* < \frac{\overline{w}}{1+\beta}$ that is relatively high and could not be obtained at Nash equilibrium. Without a public intervention, the child would exert too much LTC effort and would make a cash transfer to her parent: this equilibrium would be first-best Pareto-efficient but would lead to a net wealth for the child $y_2 = \frac{\overline{w}}{1+3\beta}$, which is too low compared to what the State wants. We note that the higher β, the lower $\frac{\overline{w}}{1+3\beta}$, so a very altruistic child could be very poor at equilibrium. To increase the child's wealth at equilibrium beyond $\frac{\overline{w}}{1+3\beta}$, the government must *tax* the child's and the parent's LTC effort. In doing this, the State gives up efficient LTC efforts.

Equations (20) and (21) refers to situations in which the State wants the child to have a relatively low net wealth $y^* < \frac{\alpha}{3+\alpha}\overline{w}$, and so a relatively high net wealth for the parent. Without a public intervention, the child's net wealth in equilibrium is too high ($y_2 = \frac{\alpha\,\overline{w}}{2+\alpha+\alpha\beta} > \frac{\alpha\,\overline{w}}{3+\alpha}$) because the parent makes her a cash transfer. If α tends to infinity, y_2 tends to $\overline{w}$ at equilibrium, which means that the child holds all the family's wealth at equilibrium and no LTC effort is exerted. To increase the parent's wealth and LTC efforts, the government must *subsidize* LTC efforts despite this leading to *inefficient* levels of efforts.

7 Conclusion

This work set out to analyze how families faced with the dependency of an older parent manage to meet the needs of their dependent parent and carry the financial burden that this situation imposes on each family member. For this purpose, we developed a model in between altruism and exchange models. Parent and child are altruistic by definition, but exchange processes appear as a property of the Nash equilibrium (they are not presupposed). Our first objective was to understand how the members of the same family – who are concerned about the well being of their loved ones – make individual decisions in terms of LTC efforts (formal for the parent and informal for the child) and cash transfers (a priori) in both directions. We observed by studying the Nash equilibrium that altruistic family members use cash transfers to reduce inequalities within the family according to their own conception of "family justice" (depending on their parameters of altruism and wealth), which can be very different from that of a regulator. We have shown that in the case of equilibrium with transfer, the LTC efforts chosen by the agents do not depend on the initial wealth of each one but on the final distribution (i.e. after transfer) of the family's wealth. Studying the formation of this equilibrium enables us to understand the origin of situations that we sometimes observe in reality, in which a parent or child voluntarily "sacrifices" him- or herself for the well-being of their loved one. The comparative static exercise on equilibrium shows us that the effect of exogenous variables could be different depending on the nature of the equilibrium. This is in particular the reason why the neutrality property (see Barro 1974) holds at equilibrium with transfer but not at equilibrium with no transfers. Interestingly, when there is no transfer at Nash equilibrium, we find that a public lump sum transfer to the child could be Pareto-improving because there is no longer a crowding-out effect (due to intra-family transfers). In this work, our conception of optimality is different from earlier literature in which the first-best is approached as the allocation maximizing the sum of individuals utilities. We retain instead a Pareto criterion and define the *set* of Pareto-optimal allocations. We find that only the Nash equilibrium with a transfer from child to parent is a Pareto-optimal allocation. We also find that the parent may over-invest in LTC effort at equilibrium. We finally show that some first-best optimal allocations (not all) and second-best allocations can be decentralized by subsidizing informal care. The question thus arise of how to subsidize a child's informal long-term effort in practice.

In our model, like in real life, informal care provided by a child corresponds to an unpaid effort resulting in a monetary loss for the child. While informal care seems to save private and public spending, it incurs significant hidden cost to carers in practice. This cost includes not only the opportunity cost of lost earnings, but also the monetary equivalent of the consequences of the physical and psychological burden experienced by the carer. Concretely, the hidden cost of informal care is now well recognized, and many countries are developing strategies to reduce it, which can be viewed as a kind of informal care subsidy. A recent report of the United Nations Economic Commission for Europe presents policy strategies implemented in the UNECE region by national or local governments to support (subsidize) informal carers. Different measures have been implemented depending on the country.

It appears from this study that the first step in public policies developed to support carers is often to improve the acknowledgement of informal carers as co-producers of LTC services. Several countries have put in place programs to raise awareness among informal carers themselves (many have never identified themselves as such), the general public, employers and social services (these programs include information campaigns, practical guides, organization of seminars in social services, government departments, or labor offices). Improving the recognition of informal carers is indeed the prerequisite for developing and implementing adequate solutions to the different challenges faced by informal carers.

One of the main challenges for carers is to combine time-consuming care responsibilities with gainful employment. This is sometimes so hard that carers decide to quit their jobs, with serious financial consequences. A first set of solutions that have been adopted in several countries consists in providing working carers with access to care leaves and flexible working arrangements to help them meet their care responsibilities while remaining in paid employment.

Informal carers need to juggle informal care not only with their jobs but also with meeting their own or other family members' needs (typically those of their own children). To help them meet this further challenge, many countries have implemented respite care services (in-home nursing, out-of-home day-care services) thanks to which informal carers can take a short-term break from their care responsibilities and recover time for other activities. To facilitate the reconciliation of informal care and personal life, it is useful to supplement the right to respit with a better access of carers to support services such as nurseries, public transports, household help or psychological help.

All the above measures must of course be accompanied by financial support to carers, in the form not only of allowances but also of a social security coverage (including health care coverage and pension contributions).

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8 Appendix

8.1 Proof of Proposition 1

Step one : The equilibrium conditions

We define the best response of the parent (e_1 and t_1) as the solution to programm $P_1(e_2, t_2)$

given below

$$\max_{e_1,t_1} \; u(w_1 - e_1 - t_1 + t_2) + h(e_1, e_2) + \alpha \; v(w_2 - e_2 - t_2 + t_1) \qquad (P_1(e_2, t_2))$$

$$st \; w_1 - e_1 - t_1 + t_2 \geq 0$$

$$e_1 \in [0, \overline{e}] \,, \; t_1 \; \in [0, \overline{w}] \,.$$

Under our assumptions, (e_1, t_1) is the solution of $P_1(e_2, t_2)$ if and only if it solves the necessary conditions of optimality of Kuhn and Tucker. Consequently, there are some non-negative multipliers $\mu_1, \sigma_e^1, \omega_e^1, \sigma_t^1, \omega_t^1$ such that

$$-u' + h_1 - \mu_1 + \sigma_e^1 - \omega_e^1 = 0 \tag{22}$$

$$-u' + \alpha v' - \mu_1 + \sigma_t^1 - \omega_t^1 = 0 \tag{23}$$

$$\mu_1(w_1 - e_1 - t_1 + t_2) = 0, \; w_1 - e_1 - t_1 + t_2 \geq 0 \tag{24}$$

$$\sigma_e^1 e_1 = \omega_e^1(\overline{e} - e_1) = 0, \; e_1 \in [0, \overline{e}] \tag{25}$$

$$\sigma_t^1 t_1 = \omega_t^1(\overline{w} - t_1) = 0, \; t_1 \in [0, \overline{w}] \,. \tag{26}$$

The same reasoning holds for the child. Her best response solves the program $P_2(e_1, t_1)$

$$\max_{e_2,t_2} \; v(w_2 - e_2 - t_2 + t_1) + \beta \; (u(w_1 - e_1 - t_1 + t_2) + h(e_1, e_2)) \qquad (P_2(e_1, t_1))$$

$$st \; w_2 - e_2 - t_2 + t_1 \geq 0$$

$$e_2 \in [0, \overline{e}] \,, \; t_2 \; \in [0, \overline{w}] \,.$$

There are therefore some non-negative multipliers $\mu_2, \sigma_e^2, \omega_e^2, \sigma_t^2, \omega_t^2$ such that the solution of $P_2(e_1, t_1)$ satisfies

$$-v' + \beta h_2 - \mu_2 + \sigma_e^2 - \omega_e^2 = 0 \tag{27}$$

$$\beta u' - v' - \mu_2 + \sigma_t^2 - \omega_t^2 = 0 \tag{28}$$

$$\mu_2(w_2 - e_2 - t_2 + t_1) = 0, \; w_2 - e_2 - t_2 + t_1 \geq 0 \tag{29}$$

$$\sigma_e^2 e_2 = \omega_e^2(\overline{e} - e_2) = 0, \; e_2 \in [0, \overline{e}] \tag{30}$$

$$\sigma_t^2 t_2 = \omega_t^2(\overline{w} - t_2) = 0, \; t_2 \in [0, \overline{w}] \,. \tag{31}$$

As a consequence (e_1, t_1, e_2, t_2) is an equilibrium of the game if and only if there are non-negative multipliers $\mu_i, \sigma_e^i, \omega_e^i, \sigma_t^i, \omega_t^i$, $i = 1, 2$ such that equations (22) to (31) hold.

Step 2 : One of the two transfers equates zero.

Suppose this is false. We then have $t_1 > 0$ and $t_2 > 0$ so $\sigma_t^1 = \sigma_t^2 = 0$. (23) implies $\alpha v' \geq u'$ and (28) implies $\beta u' \geq v'$, as a consequence we then have $\alpha\beta v' \geq \beta u' \geq v'$, so $\alpha\beta$ is such that $\alpha\beta \geq 1$, which conflicts with our assumptions, hence a contradiction.

Step 3 : All multipliers except σ_t^1 and σ_t^2 equal zero.

· We first show that $t_1 < \overline{w}$ (and $\omega_t^1 = 0$). Let us assume the opposite $t_1 = \overline{w}$. As only one transfer is positive, we then have $t_2 = 0$. The constraint according to the parent's wealth must be non-negative $(w_1 - e_1 - t_1 + t_2 \geq 0)$. It thus becomes $-w_2 - e_1 \geq 0$, which conflicts with $e_1 \geq 0$, hence a contradiction.

The same reasoning applies to the child's transfer, so in equilibrium we have $t_2 < \overline{w}$ (and $\omega_t^2 = 0$).

· We now show that $e_i < \bar{e}$ and $\omega_e^1 = \omega_e^2 = 0$ in equilibrium. As the net wealths of the parent and his child are non-negative, the equilibrium must meet the two inequalities $w_1 - e_1 - t_1 + t_2 \geq 0$ and $w_2 - e_2 - t_2 + t_1 \geq 0$. Summing these two inequalities gives $\overline{w} \geq e_1 + e_2$, which implies that $e_1 \leq \overline{w}$ and $e_2 \leq \overline{w}$ as efforts e_1 and e_2 are non-negative reals. By assumption, $\bar{e} > \overline{w}$, so no effort in equilibrium can reach the highest value of feasible effort $\bar{e}$.

· Taking into account $\omega_t^i = \omega_e^i = 0$, $i = 1, 2$, rewriting (22) (23) and (27) (28) gives

$$h_1 + \sigma_e^1 = u'(w_1 - e_1 - t_1 + t_2) + \mu_1 = \alpha v'(w_2 - e_2 - t_2 + t_1) + \sigma_t^1 \tag{32}$$

$$\beta h_2 + \sigma_e^2 = v'(w_2 - e_2 - t_2 + t_1) + \mu_2 = \beta u'(w_1 - e_1 - t_1 + t_2) + \sigma_t^2. \tag{33}$$

We observe that if $w_1 - t_1 + t_2 = 0$ in equilibrium, then $e_1 = 0$ (as the constraint means that $w_1 - t_1 + t_2 \geq e_1 \geq 0$). As $u'(0)$ tends to $+\infty$, the second equation implies that $v'(w_2 - e_2 - t_2 + t_1) + \mu_2$ tends to infinity, so either $w_2 - e_2 - t_2 + t_1 = 0$ or $\mu_2 > 0$ (i.e. $w_2 - e_2 - t_2 + t_1 = 0$). Consequently, $e_2 = \overline{w}$, hence a contradiction, as $h_2(\overline{w}, 0)$ is bounded by assumption.

Symmetric reasoning shows that $w_1 - t_1 + t_2 > 0$.

We observe that if $e_1 = 0$ in equilibrium, then $e_2 > 0$ leads to $h_1(0, e_2) \leq u'(w_1 - t_1 + t_2)$, which is impossible as $h_1(0, e_2)$ tends to $+\infty$ (a similar reasoning excludes $e_2 = 0$ and $e_1 > 0$ in equilibrium). Finally, we observe that $e_1 = 0$ and $e_2 = 0$ implies $h_1(0,0) + h_2(0,0) \leq u'(w_1 - t_1 + t_2) + \dfrac{v'(w_2 - t_2 + t_1)}{\beta}$ as $w_1 - t_1 + t_2 > 0$ and $w_2 - t_2 + t_1 > 0$, hence a contradiction, as the right-hand term goes to infinity and the left one is finite. This last point thus shows that both LTC efforts and net wealths are positive, i.e. $e_1 e_2 > 0$, and $(w_2 - e_2 - t_2 + t_1)(w_1 - e_1 - t_1 + t_2) > 0$ in equilibrium.

· The following equations then totally define the equilibrium, as stated in Proposition 1:

$$h_1 = u' = \alpha v' + \sigma_t^1$$

$$\beta h_2 = v' = \beta u' + \sigma_t^2$$

$$\sigma_t^k t_k = 0 \text{ for } k = 1, 2.$$

8.2 Proof of Proposition 2

Proposition 1 can be rewritten in the following way: (e_1, e_2, t_1, t_2) is an equilibrium of the game if and only if (e_1, e_2, t_1, t_2), $\lambda \in (0, 1)$ and $\gamma \in [\alpha\beta, 1]$ satisfy the following conditions:

$$w_1 - t_1 + t_2 = \lambda \overline{w} \tag{34}$$

$$u'(\lambda \overline{w} - e_1) = h_1(e_1, e_2) \tag{35}$$

$$v'((1 - \lambda)\overline{w} - e_2) = \beta h_2(e_1, e_2) \tag{36}$$

$$h_1(e_1, e_2) = \gamma h_2(e_1, e_2) \tag{37}$$

$$\text{either } \lambda = \lambda_0 \text{ if } \gamma \in (\alpha\beta, 1) \text{ (and } t_1 = t_2 = 0) \text{ or} \tag{38}$$

$$\lambda > \lambda_0 \text{ with } \gamma = 1 \text{ and } t_1 = 0 \text{ or } \lambda < \lambda_0 \text{ with } \gamma = \alpha\beta \text{ and } t_2 = 0. \tag{39}$$

Equations (34) and (37) provide the definitions of λ and γ.

We now prove a main result.

Lemma 1 *Given any $\lambda \in (0, 1)$, there is a unique $(e_1, e_2, \gamma) = (\widehat{e}_1(\lambda), \widehat{e}_2(\lambda), g(\lambda))$ solution of (35) and (36),(37). Moreover, $g(\lambda)$ decreases when λ increases, with* $\lim_{\lambda \to 0} g(\lambda) = +\infty$ *and* $\lim_{\lambda \to 1} g(\lambda) = 0$.

□Proof : Given λ, we observe that the two efforts (e_1, e_2) solution of (35) and (36) are also the solution of the following program:

$$\max_{e_1 \in [0, \lambda \overline{w}], e_2 \in [0, (1-\lambda)\overline{w}]} v((1 - \lambda)\overline{w} - e_2) + \beta \ (u(\lambda \overline{w} - e_1) + h(e_1, e_2)) \, .$$

As V is a concave function of (e_1, e_2), the solution of this program exists, is interior, unique and continuous with respect to λ, so the two equations (35) and (36) have a solution.

Differentiating (35) and (36) with respect to λ gives

$$u''\overline{w} = (u'' + h_{11})\widehat{e}_1'(\lambda) + h_{12}\widehat{e}_2'(\lambda) \tag{40}$$

$$-v''\overline{w} = \beta(h_{21}\widehat{e}_1'(\lambda) + (v'' + \beta h_{22})\widehat{e}_2'(\lambda); \tag{41}$$

and solving this system leads to

$$\begin{aligned}\widehat{e}_1'(\lambda) &= \frac{\overline{w}\ \left\{u''(v'' + \beta h_{22}) + v'' h_{12}\right\}}{(u'' + h_{11})(v'' + \beta h_{22}) - \beta(h_{12})^2} > 0 \\ \widehat{e}_2'(\lambda) &= \frac{-\overline{w}\ \left\{(u'' + h_{11})v'' + \beta h_{12}u''\right\}}{(u'' + h_{11})(v'' + \beta h_{22}) - \beta(h_{12})^2} < 0.\end{aligned}$$

We deduce that $g(\lambda)$ (meeting (37)) is strictly decreasing in λ, as we have:

$$\begin{aligned}\frac{d\ln g}{d\lambda} &= \frac{d\ln h_1}{d\lambda} - \frac{d\ln h_2}{d\lambda} = (\frac{h_{11}}{h_1} - \frac{h_{21}}{h_2})\widehat{e}_1'(\lambda) + (\frac{h_{12}}{h_1} - \frac{h_{22}}{h_2})\widehat{e}_2'(\lambda) \\ &= \overline{w}\frac{\left(\frac{h_{11}}{h_1} + \frac{h_{22}}{h_2}\right)u''v'' + (\frac{h_{11}h_{22} - (h_{12})^2}{h_1})(\frac{u''\beta}{h_1} + \frac{v''}{h_2})}{(u'' + h_{11})(v'' + \beta h_{22}) - \beta(h_{12})^2} < 0.\end{aligned}$$

We now observe that :

$$\lim_{\lambda \to 0} g(\lambda) = +\infty \text{ and } \lim_{\lambda \to 1} g(\lambda) = 0.$$

We recall that :

$$g(\lambda) = \frac{h_1(\widehat{e}_1(\lambda), \widehat{e}_2(\lambda))}{h_2(\widehat{e}_1(\lambda), \widehat{e}_2(\lambda))} = \beta\frac{u'(\lambda\overline{w} - \widehat{e}_1(\lambda))}{v'((1-\lambda)\overline{w} - \widehat{e}_2(\lambda))}.$$

We first assume that λ tends to zero. $\lambda\overline{w}$ and $\widehat{e}_1(\lambda)$ thus tend to zero, so $u'(\lambda\overline{w} - \widehat{e}_1(\lambda))$ tends to $+\infty$. Consequently, if $g(\lambda)$ is bounded, $\widehat{e}_2(\lambda)$ must tend to $(1-\lambda)\overline{w}$ when λ tends to zero. But by assumption $\lim_{e_1 \to 0} \dfrac{h_1(e_1, \overline{w})}{h_2(e_1, \overline{w})} = +\infty$, hence a contradiction. We now assume that λ tends to 1 so $\widehat{e}_2(\lambda)$ tends to zero and $v'((1-\lambda)\overline{w} - \widehat{e}_2(\lambda))$ tends to $+\infty$. If $g(\lambda)$ does not tend to zero, then $\lambda\overline{w} - \widehat{e}_1(\lambda)$ tends to zero, that is $\widehat{e}_1(\lambda)$ tends to $\overline{w}$. But by assumption $\lim_{e_2 \to 0} \dfrac{h_1(\overline{w}, e_2)}{h_2(\overline{w}, e_2)} = +\infty$, hence a contradiction. □

The equilibrium can thus be described in the following terms. In equilibrium, there exists λ in $[g^{-1}(1), g^{-1}(\alpha\beta)]$ such that $e_i = \widehat{e}_i(\lambda)$, $i = 1, 2$ and the transfers are defined by equation (34) $w_1 - t_1 + t_2 = \lambda\overline{w}$. What follows puts (38) and (39) in other terms:

if $\lambda_0 < g^{-1}(1)$, then the equilibrium is obtained for $\gamma = 1$ (the child makes a positive transfer to his parent),

if $\lambda_0 \in [g^{-1}(1), g^{-1}(\alpha\beta)]$, then the equilibrium is obtained without transfer for a value of γ such that $\lambda_0 = \lambda^*(\gamma)$, and if $\lambda_0 > g^{-1}(\alpha\beta)$, then the equilibrium is obtained for $\gamma = \alpha\beta$ (and the parent makes a positive transfer to his child).

8.3 Proof of Proposition 3

In the equilibrium without transfer, the equilibrium equations are given by

$$e_1 + \varphi_1(h_1) = w_1$$
$$e_2 + \varphi_2(\beta h_2) = w_2,$$

where $\varphi_1(u(y)) = y$ and $\varphi_2(v(y)) = y$.

We differentiate the system with respect to all variations of parameters, giving

$$\begin{aligned} (1+\varphi_1' h_{11})de_1 + \varphi_1' h_{12} de_2 &= dw_1 \\ \beta\varphi_2' h_{21} de_1 + (1+\beta\varphi_2' h_{22})\, de_2 &= dw_2 - \varphi_2' h_2 d\beta. \end{aligned}$$

We thus obtain, once the system has been solved:

$$\begin{aligned} de_1 &= \frac{(1+\beta\varphi_2' h_{22})\, dw_1 - \varphi_1' h_{12}(dw_2 - \varphi_2' h_2 d\beta)}{(1+\beta\varphi_2' h_{22})\,(1+\varphi_1' h_{11}) - \beta\varphi_2'(h_{21})^2} \\ de_2 &= \frac{-\beta\varphi_2' h_{21} dw_1 + (dw_2 - \varphi_2' h_2 d\beta)\,(1+\varphi_1' h_{11})}{(1+\beta\varphi_2' h_{22})\,(1+\varphi_1' h_{11}) - \beta\varphi_2'(h_{21})^2}. \end{aligned}$$

Observe that the denominator is positive. This provides Table 2 presented in Proposition 3.

8.4 Proof of Proposition 4

To lighten the proof, we define $\Delta = t_1 - t_2$ and $\rho = \alpha\beta$ si $t_1 > 0$ ou $\rho = 1$ si $t_2 > 0$. The equilibrium conditions satisfy:

$$\Delta + e_1 + \varphi_1(h_1) = w_1$$
$$e_1 + e_2 + \varphi_2(\beta h_2) + \varphi_1(h_1) = \overline{w}$$
$$h_1 - \rho h_2 = 0.$$

We first differentiate the system, then calculate de_1 and de_2 using the last two equations. Substitution in the first one gives the value of $d\Delta$.

$$\begin{aligned} d\Delta + (1+\varphi_1' h_{11})de_1 + \varphi_1' k_{12} de_2 &= dw_1 \\ (1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})\, de_1 + (1+\beta\varphi_2' h_{22} + \varphi_1' h_{12})\, de_2 &= -\varphi_2' h_2 d\beta + d\overline{w} \\ (h_{11} - \rho h_{21}) de_1 + (h_{12} - \rho h_{22}) de_2 &= h_2 d\rho. \end{aligned}$$

Cramer's rule gives:

$$\begin{aligned} de_1 &= \frac{(-\varphi_2' h_2 d\beta + d\overline{w})(h_{12} - \rho h_{22}) - h_2 d\rho\,(1+\beta\varphi_2' h_{22} + \varphi_1' h_{12})}{(1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})(h_{12} - \rho h_{22}) - (h_{11} - \rho h_{21})\left(1+\widehat{\beta}\varphi_2' h_{22} + \varphi_1' h_{12}\right)} \\ de_2 &= \frac{(1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})\, h_2 d\rho - (h_{11} - \rho h_{21})(-\varphi_2' h_2 d\beta + d\overline{w})}{(1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})(h_{12} - \rho h_{22}) - (h_{11} - \rho h_{21})(1+\beta\varphi_2' h_{22} + \varphi_1' h_{12})}. \end{aligned}$$

We observe that the denominator (D) is positive. Rewriting this gives

$$D = h_{12} - h_{11} - \beta\varphi_2'(h_{11}h_{22} - (h_{21})^2) + \rho\left\{h_{12} - h_{22} - \varphi_1'(h_{11}h_{22} - (h_{21})^2)\right\} > 0.$$

We thus obtain the sign of de_1 and de_2, noting that $d\rho = \beta d\alpha + \alpha d\beta$ if $t_1 > 0$ and $d\rho = 0$ if $t_2 > 0$. This is summarized in the table below.

$\Delta \neq 0$	de_1	de_2
$d\overline{w}$	$h_{12} - \rho h_{22} > 0$	$\rho h_{21} - h_{11} > 0$
$d\beta$ si $t_2 > 0$	$-\varphi_2'(h_{12} - \rho h_{22}) > 0$	$(h_{11} - \rho h_{21})\varphi_2' > 0$
$d\alpha$ si $t_1 > 0$	$-(1+\beta\varphi_2' h_{22} + \varphi_1' h_{12})$	$(1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})$
$d\beta$ si $t_1 > 0$	$-(\alpha + \alpha\varphi_1' h_{12} + \varphi_v' h_{12})$	$(1+\varphi_1' h_{11})\,\alpha + \varphi_2' h_{11} > 0$

We finally obtain $d\Delta$ using the first equation of derivatives ;

$$\begin{aligned} D\ d\Delta &= Ddw_1 - (1+\varphi_1' h_{11})Dde_1 - \varphi_1' h_{12} Dde_2 \\ &= (1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})(h_{12} - \rho h_{22})dw_1 - (h_{11} - \rho h_{21})(1+\beta\varphi_2' h_{22} + \varphi_1' h_{12})\, dw_1 \\ &\quad -(1+\varphi_1' h_{11})(-\varphi_2' h_2 d\beta + d\overline{w})(h_{12} - \rho h_{22}) + h_2 d\rho\,(1+\beta\varphi_2' h_{22} + \varphi_1' h_{12})(1+\varphi_1' h_{11}) \\ &\quad -\varphi_1' h_{12}\,(1+\beta\varphi_2' h_{21} + \varphi_1' h_{11})\, h_2 d\rho + \varphi_1' h_{12}(h_{11} - \rho h_{21})(-\varphi_2' h_2 d\beta + d\overline{w})\,; \end{aligned}$$

which gives after simplification

$$\begin{aligned}
Dd\Delta &= \left\{\beta\varphi_2'((h_{12})^2 - h_{11}h_{22}) - (h_{11} - \rho h_{21})\right\} dw_1 \\
&+(-\varphi_2' h_2 d\beta + dw_2)\left\{-(h_{12} - \rho h_{22}) + \rho\varphi_1'(h_{11}h_{22} - (h_{12})^2)\right\} \\
&+h_2 d\rho\left\{1 + \varphi_1' h_{11} + \beta\varphi_2' h_{22} + \beta\varphi_1'\varphi_2'((h_{11}h_{22} - (h_{12})^2)\right\};
\end{aligned}$$

yielding the following table of signs:

$\Delta \neq 0$	$d\Delta$
dw_1	> 0
dw_2	< 0
$d\beta$ si $t_2 > 0$	< 0
$d\alpha$ si $t_1 > 0$	> 0
$d\beta$ si $t_1 > 0$	$\alpha(1 + \varphi_1' h_{11}) + \varphi_2' h_{12}$.

8.5 Proof of Proposition 5

Let us assume a variation in the LTC effort of respectively the parent and the child de_1 and de_2, such that the total LTC expense remains constant, $de_1 + de_2 = 0$. This change increases the parent's health status if and only if

$$h_1 de_1 + h_2 de_2 \geq 0 \Leftrightarrow (\frac{h_1}{h_2} - 1)\ de_1 \geq 0 \text{ since } de_1 = -de_2.$$

Since the marginal rate of technical substitution of e_1 for e_2 is less than 1 in equilibrium if the child makes no transfer to her parent, the parent's health status increases if $de_1 < 0$, that is if the parent decreases his care effort and the child's increases hers. An increase in the transfer paid by the parent to the child equal to the child's informal care effort $dt_1 = de_2 = -de_1$ is possible since this increase leaves the parent's and the child's net wealth unchanged ($dy_1 = -dt_1 - de_1 = 0$ and $dy_2 = dt_1 - de_2 = 0$). Consequently, this change increases the parent's well-being without decreasing the child's well-being. It is thus a Pareto-improving change compared to the initial situation.

8.6 Proof of Proposition 6

Recall that a first-best allocation (e_1, e_2, y_1, y_2) is defined by :

$$\begin{aligned} y_1 + y_2 + e_1 + e_2 &= \overline{w} \\ h_1(e_1, e_2) - h_2(e_1, e_2) &= 0 \\ u'(y_1) - h_1(e_1, e_2) &= 0. \end{aligned}$$

Differentiating this system with respect to y_2 gives:

$$dy_1 + de_1 + de_2 = -dy_2$$

$$(h_{11} - h_{21})de_1 + (h_{12} - h_{22})de_2 = 0$$

$$u''(y_1)dy_1 - h_{11}de_1 - h_{12}de_2 = 0,$$

so that we finally obtain

$$dy_1 = \frac{-h_{11}\ dy_2}{(u'' + h_{11}) + (u'' + h_{12})\dfrac{h_{11} - h_{21}}{h_{22} - h_{12}}} < 0.$$

We now consider the first-best allocation $(e_1^*(y_2), e_2^*(y_2), y_1^*(y_2))$ expressed as a function of $y_2 \in (0, \overline{w})$. Observe that a first-best allocation can be decentralized if and only if (16) holds, that is:

$$A(y_2) \equiv \frac{u'(y_1(y_2))}{v'(y_2)} \in \left[\alpha, \frac{1}{\beta}\right]. \tag{42}$$

Differentiating $A(.)$ easily shows that A is an increasing function of y_2. Moreover, A tends to 0 when y_2 tends to 0 and A tends to $+\infty$ when y_2 tends to $\overline{w}$. Consequently, there are two thresholds $\underline{y}^*$ and $\overline{y}^*$ in between 0 and $\overline{w}$ such that

$$A(\underline{y}^*) = \alpha \text{ and } A(\overline{y}^*) = \frac{1}{\beta}.$$

8.7 Second-best allocations in the numerical example

The program definig second-best allocations becomes:

$$\max_{e_i,\tau_i,y_i,i=1,2} \quad \ln y_1 + \ln e_1 + \ln e_2$$

$$y_2 \geq y^*$$

$$y_1 + y_2 + e_1 + e_2 = \overline{w}$$

$$y_2 - \alpha y_1 \geq 0.$$

$$y_1 - \beta y_2 \geq 0.$$

We observe that this program is convex, and the set of constraints leads to $y^*(1+\beta) < \overline{w}$. Let $\mathcal{L}$ be the Lagrange function given below :

$$\mathcal{L} = \ln y_1 + \ln e_1 + \ln e_2 + \mu_1(y_2 - y^*) + \mu_2(\overline{w} - \ y_1 - y_2 - e_1 - e_2) + \mu_3(y_2 - \alpha y_1) + \mu_4(y_1 - \beta y_2).$$

The necessary optimality conditions are as follows:

$$\frac{1}{y_1} - \mu_2 - \alpha\mu_3 + \mu_4 = 0$$

$$\mu_1 - \mu_2 + \mu_3 - \beta\mu_4 = 0$$

$$\frac{1}{e_1} - \mu_2 = 0$$

$$\frac{1}{e_2} - \mu_2 = 0$$

$$\mu_1(y_2 - y^*) = 0 \text{ and } \mu_1 \geq 0 \text{ and } (y_2 - y^*) \geq 0$$

$$\mu_2(\overline{w} - \ y_1 - y_2 - e_1 - e_2) = 0 \text{ and } y_1 + y_2 + e_1 + e_2 = \overline{w} = 0$$

$$\mu_3(y_2 - \alpha y_1) = 0 \text{ and } \mu_3 \geq 0, y_2 - \alpha y_1 \geq 0$$

$$\mu_4(y_1 - \beta y_2) = 0 \text{ and } \mu_4 \geq 0,\ y_1 - \beta y_2 \geq 0.$$

We first consider that $\mu_3 = \mu_4 = 0$. We thus obtain from the CNO $y_2 = y^*$ and $e_1 = e_2 = y_1 = \frac{\overline{w} - y^*}{3} = \mu_1 = \mu_2$. These solutions are feasible on condition that $y^* - \alpha\frac{\overline{w} - y^*}{3} \geq 0$ and $\frac{\overline{w} - y^*}{3} - \beta y^* \geq 0$. All these conditions correspond to equation (18).

We now consider that $\mu_3 = 0$ and $\mu_4 > 0$. This gives (19) as:

$$\mu_4 = -\frac{1}{\beta y^*} + \frac{2}{\overline{w} - y^*(1+\beta)} \geq 0 \Leftrightarrow (1+3\beta)y^* \geq \overline{w}$$
$$\mu_1 = \mu_2 + \beta\mu_4 > 0 \text{ and } \mu_2 = \frac{2}{\overline{w} - y^*(1+\beta)}.$$

To obtain (20), we consider that $\mu_3 > 0$, $\mu_4 = 0$ and $\mu_1 > 0$. We then obtain from the CNO:

$$\mu_3 = \frac{1}{y^*} - \frac{2}{\alpha\overline{w} - y^*(1+\alpha)} > 0 \Leftrightarrow \alpha\overline{w} > y^*(3+\alpha)$$
$$\mu_1 = \frac{2(1+\alpha)}{\alpha\overline{w} - y^*(1+\alpha)} - \frac{1}{y^*} \geq 0 \Leftrightarrow 3y^*(1+\alpha) \geq \alpha\overline{w}$$
$$\mu_2 = \frac{2\alpha}{\alpha\overline{w} - y^*(1+\alpha)}.$$

Finally, to obtain (21), we consider that $\mu_3 > 0$, $\mu_4 = 0$ and $\mu_1 = 0$. We then obtain from the CNO:

$$y_1 = \frac{\overline{w}}{3(1+\alpha)}$$
$$\frac{1}{e_2} = \frac{1}{e_1} = \mu_2 = \mu_3 = \frac{3}{\overline{w}}$$
$$y_2 = \frac{\overline{w}\alpha}{3(1+\alpha)} \geq y^*.$$